

## 2-Filteredness and The Point of Every Galois Topos

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Received: 18 December 2007 / Accepted: 1 May 2008  
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**Abstract** A connected locally connected topos is a *Galois topos* if the Galois objects generate the topos. We show that the full subcategory of Galois objects in any connected locally connected topos is an inversely 2-filtered 2-category, and as an application of the construction of 2-filtered bi-limits of topos, we show that every Galois topos has a point.

**Keyword** Galois topos 2-filtered

**Mathematics Subject Classification (2000)** 18B25

### Introduction

Galois topoi (Definition 1.5) arise in Grothendieck's Galois theory of locally connected topoi. They are a special kind of atomic topoi. It is well known that atomic topoi may be pointless [5], however, in this paper we show that any Galois topos has points.

We show how the full subcategory of Galois objects (Definition 1.2) in any connected locally connected topos  $\mathcal{E}$  has an structure of 2-filtered 2-category (in the sense of [3]). Then we show that the assignment, to each Galois object  $A$ , of the category  $\mathcal{D}_A$  of connected locally constant objects trivialized by  $A$  (Definition 3.1), determines a 2-functor into the category of categories. Furthermore, this 2-system becomes a pointed 2-system of pointed sites (considering the topology in which each single arrow is a cover). By the results on 2-filtered bi-limits of topoi (Dubuc and Yuhjtman, submitted for publication), it follows that, if  $\mathcal{E}$  is a Galois topos, then it is the bi-limit of this system, and thus, it has a point.

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## Context

Throughout this paper  $\mathcal{S} = \mathit{Sets}$  denotes the topos of sets. All topoi  $\mathcal{E}$  are assumed to be Grothendieck topoi (over  $\mathcal{S}$ ), the structure map will be denoted by  $\gamma : \mathcal{E} \rightarrow \mathcal{S}$  in all cases.

## 1 Galois Topoi and the 2-Filtered 2-Category of Galois Objects

We recall now the definition of Galois object in a topos. The original definition of Galois object given in [4] was relative to a surjective point of the topos:

**Definition 1.1** Let  $\mathcal{E} \xrightarrow{\gamma} \mathcal{S}$  be a topos furnished with a surjective point, that is, a geometric morphism  $\mathcal{S} \xrightarrow{p} \mathcal{E}$  whose inverse image functor reflects isomorphisms. Then, an object  $A$  is a Galois object if:

- i) There exists  $a \in p^*A$  such that the map  $\mathit{Aut}(A) \xrightarrow{a^*} p^*A$ , defined by  $a^*(h) = p^*(h)(a)$  is a bijection (the same holds then for any other  $b \in p^*A$ ).
- ii)  $A$  is connected and  $A \rightarrow 1$  is epimorphic.

Notice that in the context of the classical Galois theory (Artin's interpretation) this definition coincides with the definition of normal extension.

It is easy to check that in the presence of a (surjective) point the following unpointed definition is equivalent.

**Definition 1.2** An object  $A$  in a topos  $\gamma : \mathcal{E} \rightarrow \mathcal{S}$  is a Galois object if:

- i) The canonical map  $A \times \gamma^* \mathit{Aut}(A) \rightarrow A \times A$ , described by  $(a, h) \mapsto (a, h(a))$ , is an isomorphism.
- ii)  $A$  is connected and  $A \rightarrow 1$  is epimorphic.

In particular,  $A$  is a connected locally constant object.

**Proposition 1.3** Let  $A$  be a Galois object in a locally connected topos  $\mathcal{E} \xrightarrow{\gamma} \mathcal{S}$ , and let  $X$  be any locally constant object such that there exists an epimorphism  $A \xrightarrow{e} X$ . Then, the canonical map  $A \times \gamma^*[A, X] \rightarrow A \times X$ , described by  $(a, f) \mapsto (a, f(a))$ , is an isomorphism. In particular,  $X$  is connected and it is split by the cover  $A \rightarrow 1$ .

*Proof* Consider the following commutative diagram:

$$\begin{array}{ccc}
 A \times \gamma^* \mathit{Aut}(A) & \xrightarrow{\cong} & A \times A \\
 \downarrow A \times \gamma^* e_* & & \downarrow A \times e \\
 A \times \gamma^*[A, X] & \longrightarrow & A \times X
 \end{array}$$

This shows that the map (bottom row) is an epimorphism. To see that it is also a monomorphism, let  $Z \begin{matrix} \xrightarrow{s} \\ \xrightarrow{t} \end{matrix} A \times \gamma^*[A, X]$  (with  $Z \neq \emptyset$  and connected) be a pair of maps which become equal into  $A \times X$ . The maps  $s$  and  $t$  are given by pairs  $(u, f)$  and  $(v, g)$ , with  $Z \begin{matrix} \xrightarrow{u} \\ \xrightarrow{v} \end{matrix} A$  and  $A \begin{matrix} \xrightarrow{f} \\ \xrightarrow{g} \end{matrix} X$ , such that  $u = v$  and  $f \circ u = g \circ v$ . Then,  $Y = \text{Equalizer}(f, g) \neq \emptyset$ . But  $Y$  is locally constant and  $Y \subset A$ . It follows that  $Y = A$ , that is,  $f = g$ . Since connected objects generate the topos, this finishes the proof.  $\square$

Given any two Galois objects  $A, B$ , in a connected locally connected topos  $\mathcal{E}$ , any connected component of the product  $A \times B$  is a connected locally constant object. It follows from the existence of Galois closure (see for example [2] A.1.4) that there is a Galois object  $C$  and morphisms  $C \rightarrow A, C \rightarrow B$ . The full subcategory  $\mathcal{A}$  of Galois objects fails to be (inversely) filtered because, clearly, different morphisms  $A \begin{matrix} \xrightarrow{u} \\ \xrightarrow{v} \end{matrix} B$  between Galois objects cannot be equalized from a Galois object  $C \rightarrow A$  unless they are already equal. However, we have:

**Proposition 1.4** *The category  $\mathcal{A}$  of Galois objects in a connected locally connected topos becomes a (inversely) 2-filtered 2-category (in the sense of [3]) by adding a*

*formal 2-cell*  $A \begin{matrix} \xrightarrow{u} \\ \Downarrow \theta_{vu} \\ \xrightarrow{v} \end{matrix} B$  *between any two morphisms, satisfying the following*

*equations:*

$$\theta_{uu} = id, \quad \theta_{wv} \circ \theta_{vu} = \theta_{wu}, \quad (\text{thus } \theta_{uv} = \theta_{vu}^{-1}), \quad \theta_{vu} \theta_{sr} = \theta_{svru}$$

*Proof* the proof is very easy, we let the interested reader look at the definition of 2-filtered 2-category given in [3] and verify the assertion.

After Grothendieck’s “Categories Galoisiennes” of [4] and Moerdijk’s “Galois Topos” of [6], we state the following definition:

**Definition 1.5** A Galois Topos is a connected locally connected topos generated by its Galois objects, or, equivalently, such that any connected object is covered by a Galois object.

*Remark* In Grothendieck’s and Moerdiejk’s definition it is required the topos to be pointed. Notice that Definition 1.5 above does not have this requirement.

Since Galois objects are connected locally constant objects, it follows that Galois topoi are generated by locally constant objects. On the other hand, the existence of Galois closure (see for example [2] A.1.4) shows that any such topos is a Galois topos.

Thus, a connected topos is a Galois topos if and only if it is generated by its connected locally constant objects. It follows:

**Proposition 1.6** Any Galois topos is a connected atomic topos; that is, is a connected locally connected boolean topos.

## 2 Galois Topoi as Filtered Bi-Limits of Topoi with Points

Consider now a connected locally connected topos  $\mathcal{E}$ , and let  $\mathcal{C}$  be a full subcategory of connected generators. Let  $A \in \mathcal{C}$  be a Galois object. We denote by  $\mathcal{C}_A \subset \mathcal{C}$  the full subcategory whose objects are the  $X \in \mathcal{C}$  below  $A$  (remark that this is not the comma category  $(A, \mathcal{C})$ ). If there is a morphism  $A \rightarrow B$  between Galois objects, clearly  $\mathcal{C}_B \subset \mathcal{C}_A$ , and if  $\mathcal{E}$  is a Galois topos, by definition the category  $\mathcal{C}$  is the filtered union:

$$\cdots \quad \mathcal{C}_B \hookrightarrow \mathcal{C}_A \hookrightarrow \cdots \hookrightarrow \mathcal{C}$$

The topos  $\mathcal{E}$  is the topos of sheaves for the canonical topology on  $\mathcal{C}$ , and each  $\mathcal{C}_A$  is itself a site with this topology. It follows from the theory of filtered inverse bi-limits of topos ([1] Expose VI) that, if  $\mathcal{E}_A$  is the topos of sheaves on  $\mathcal{C}_A$ , then the topos  $\mathcal{E}$  is an inverse bi-limit of topos:

$$\cdots \quad \mathcal{E}_B \longleftarrow \mathcal{E}_A \longleftarrow \cdots \longleftarrow \mathcal{E}$$

The representable functor  $\mathcal{C}_A \xrightarrow{[A, -]} \mathcal{S}$  is a point of the site, thus the topos  $\mathcal{E}_A$  are all pointed topos. However, a point for the site  $\mathcal{C}$  is equivalent to a simultaneous choice of points for each  $A$  commuting with all the inclusions  $\mathcal{C}_B \subset \mathcal{C}_A$ . That is, an element of the inverse limit of sets:

$$\cdots \quad \text{Points}(\mathcal{C}_B) \longleftarrow \text{Points}(\mathcal{C}_A) \longleftarrow \cdots \longleftarrow \text{Points}(\mathcal{C})$$

which, a priori, may be empty.

## 3 Galois Topoi as Pointed 2-Filtered Bi-Limits of Pointed Topoi

We shall consider next a different category associated to any Galois object.

**Definition 3.1** Let  $\mathcal{E} \xrightarrow{\gamma} \mathcal{S}$  be a connected locally connected topos, let  $\mathcal{C}$  be the subcategory of connected objects, and let  $A \in \mathcal{C}$  be any Galois object. The category  $\mathcal{D}_A$  is defined as the bi-pullback of categories:

$$\begin{array}{ccc} \mathcal{D}_A & \longrightarrow & \mathcal{C} \\ \downarrow & & \downarrow^{A \times (-)} \\ \mathcal{S} & \xrightarrow{\gamma^*} & \mathcal{E}/_A \end{array}$$

The objects and arrows of  $\mathcal{D}_A$  can be described as follows (where  $\pi_1$  denotes the first projection):

$$Ob : \text{triples } (X, S, \sigma), X \in \mathcal{C}, \emptyset \neq S \in \mathcal{S}, (\pi_1, \sigma) : A \times \gamma^* S \xrightarrow{\cong} A \times X$$

$$Arr : (X, S, \sigma) \rightarrow (Y, T, \xi) : X \xrightarrow{f} Y, S \xrightarrow{\eta} T \mid \xi(a, \eta(s)) = f(\sigma(a, s)).$$

Given any  $(X, S, \sigma) \in \mathcal{D}_A$ , since  $S \neq \emptyset$  it follows that  $[A, X] \neq \emptyset$ . Thus,  $X \in \mathcal{C}_A$ . Furthermore, since  $X$  is connected and locally constant, any map  $A \rightarrow X$  is an epimorphism. In fact, for the same reason, any map  $X \rightarrow Y$  is an epimorphism. It follows that so is  $A \times \gamma^*(\eta)$ . Since  $A$  is connected, it follows that  $\eta$  is also an epimorphism, thus a surjective function of sets.

*Remark 3.2* Given any arrow  $(X, S, \sigma) \xrightarrow{f, \eta} (Y, T, \xi)$ ,  $f$  is an epimorphism and  $\eta$  a surjective function.

**Proposition 3.3** *The functor  $\mathcal{D}_A \xrightarrow{\sim} \mathcal{C}_A, (X, S, \sigma) \mapsto X$ , is an equivalence of categories.  $\mathcal{D}_A$  has a site structure such that any single arrow is a cover. The induced morphism  $\mathcal{E}_A \xrightarrow{\sim} \mathcal{P}_A$  is an equivalence (where  $\mathcal{P}_A$  denotes the topos of sheaves on  $\mathcal{D}_A$ ).*

*Proof* Just by definition of connected object it immediately follows that given  $(X, S, \sigma), (Y, T, \xi)$ , and an arrow  $X \xrightarrow{f} Y$ , there exists a unique  $S \xrightarrow{\eta} T$  which determines an arrow  $(f, \eta)$  in  $\mathcal{D}_A$ . Thus, the functor is full and faithful. That it is essentially surjective follows by Proposition 1.3. The second assertion is clear (consider Remark 3.2). □

**Proposition 3.4**

- a) *The functor  $\mathcal{D}_A \xrightarrow{p_A^*} \mathcal{S}, p_A^*(X, S, \sigma) = S$ , determines a point of the site. This point is naturally isomorphic to the representable functor  $[A, -]$  under the equivalence  $\mathcal{D}_A \xrightarrow{\sim} \mathcal{C}_A$ .*
- b) *A morphism between Galois objects  $A \xrightarrow{u} B$  determines a morphism of sites  $\mathcal{D}_B \xrightarrow{u^*} \mathcal{D}_A$  commuting with the points  $p_A^* \circ u^* = p_B^*$ .*
- c) *Given any two morphisms between Galois objects  $A \xrightarrow[u]{u} B$ , there is a canon-*

*ical natural transformation* 
$$\mathcal{D}_A \begin{array}{c} \xrightarrow{u^*} \\ \Downarrow \theta_{vu} \\ \xrightarrow{v^*} \end{array} \mathcal{D}_B$$
 *satisfying the equations in*

*Definition 1.4*

d) The following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{C}_B & \hookrightarrow & \mathcal{C}_B \\
 \cong \uparrow & & \cong \uparrow \\
 \mathcal{D}_A & \xrightarrow{u^*} & \mathcal{D}_B \\
 & \Downarrow \theta_{vu} & \\
 & \xrightarrow{v^*} & 
 \end{array}$$

*Proof*

- a) It follows from Remark 3.2 and the fact that  $A$  is connected. Furthermore, we know there exists an epimorphism  $A \rightarrow X$ . The natural bijection  $[A, X] \cong S$  follows in the same way as item c) below (recall Proposition 1.3).
- b) Follows by the universal property of bi-pullbacks. Given  $A \xrightarrow{u} B$ , an explicit construction of  $u^*$  is the following:  $(X, S, \sigma) = u^*(Y, T, \xi)$ ,  $X = Y$ ,  $S = T$ , and  $\sigma$  is the map uniquely determined by the equation  $\sigma(s, a) = \xi(s, u(a))$ .
- c) Consider the description in b) and the following diagram:

$$\begin{array}{ccccc}
 u^*(Y, T, \xi) : & & A \times \gamma^* S & \xrightarrow{\cong} & A \times X \\
 \downarrow \theta_{uu} & & \downarrow A \times \gamma^*(\eta) & & \downarrow id \\
 v^*(Y, T, \xi) : & & A \times \gamma^* S & \xrightarrow{\cong} & A \times X
 \end{array}$$

By definition of connected object, there exists a unique  $S \xrightarrow{\eta} S$  making the square commutative. Define  $\theta_{vu} = (id_X, \eta)$ . Clearly, the equations hold by uniqueness.

Finally, d) is clear by definition of  $u^*$ ,  $v^*$  and  $\theta_{vu}$  □

### 4 Every Galois Topos $\mathcal{E}$ has a Point

It follows from Proposition 3.4 b) and c) that the assignment of the site  $\mathcal{D}_A$  to a Galois object  $A$  determines a 2-filtered 2-system of categories which has a bi-colimit that we denote  $\mathcal{D}$  (see [3]).

$$\dots \quad \mathcal{D}_B \quad \begin{array}{c} \xrightarrow{u} \\ \Downarrow \theta_{vu} \\ \xrightarrow{v} \end{array} \quad \mathcal{D}_A \quad \begin{array}{c} \longrightarrow \\ \longrightarrow \end{array} \quad \dots \quad \longrightarrow \quad \mathcal{D}$$

It follows from the results in Dubuc and Yuhjtman (submitted for publication) that the site structures in the categories  $\mathcal{D}_A$  determine a site structure on  $\mathcal{D}$  in such a way that we have an inverse bi-limit of the topoi of sheaves:

$$\dots \quad \mathcal{P}_B \begin{array}{c} \xleftarrow{v} \\ \Downarrow \theta_{vu} \\ \xleftarrow{u} \end{array} \mathcal{P}_A \xleftarrow{\quad} \dots \xleftarrow{\quad} \mathcal{P}$$

where the topos  $\mathcal{P}$  is the topos of sheaves on the site  $\mathcal{D}$ .

From Proposition 3.4 a) it follows that all the topoi  $\mathcal{P}_A$  are pointed, and from b) it follows that in this case these points induce a point  $\mathcal{S} \rightarrow \mathcal{P}$  of the bi-limit topos  $\mathcal{P}$ . Finally, from Propositions 3.4 d) and 3.3, it follows that the topoi  $\mathcal{E}$  and  $\mathcal{P}$  are equivalent topoi,  $\mathcal{E} \xrightarrow{\sim} \mathcal{P}$ . Thus,  $\mathcal{E}$  has a point  $\mathcal{S} \rightarrow \mathcal{E}$  determined by any inverse equivalence and the point of  $\mathcal{P}$ .

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